Self-Maintained Coherent Oscillations
in Dense Neutrino Gases

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Abstract

We present analytical solutions to the nonlinear equations describing the behavior of a gas of neutrinos with two flavors. Self-maintained coherent flavor oscillations are shown to occur when the gas density exceeds a critical value determined by the neutrino masses and the mean neutrino energy in the gas. Similar oscillations may have occurred in the early Universe.

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I. Introduction

The properties of neutrinos passing through matter have attracted much attention in the past few years. The reason is that the large electron-lepton number of matter results in flavor-dependent neutrino propagation and, at suitable densities, enhanced neutrino flavor oscillation [1]. The effect can be attributed primarily to neutrino-electron forward scattering through $W^{\pm}$ exchange. Among the consequences is a possible resolution of the solar neutrino problem [2].

Another situation with lepton imbalance is the early Universe [3] at a temperature below 1 MeV, where neutrino propagation is affected by the excess of electrons over positrons. Some implications for the early Universe of neutrino oscillations enhanced by $W^{\pm}$ exchange have been considered in [4]. In this scenario, however, the self-interactions of neutrinos through Z^0 exchange cannot be neglected because the neutrino density is relatively high. Indeed, numerical simulations of the early Universe show that neutrino properties can be significantly modified by these effects [5, 6, 7]. In the parameter region of [6], there is even a novel neutrino-flavor conversion mechanism that is different from the MSW effect.

The neutrino self-interactions are nonlinear. In a general context, they must be studied numerically. Moreover, in the early Universe, neutrino behavior is controlled by a combination of factors. In addition to the electron-positron imbalance and the presence of the neutrino gas itself, these include other effects such as the expansion rate of the Universe. In this paper, we eliminate these additional complications by considering a simplified situation consisting of a homogeneous gas of self-interacting neutrinos in a box of fixed volume $\mathcal{V}$, with no other leptons present. Our goal is to obtain results that are analytical but that nonetheless describe nonlinear features of neutrino behavior.

We consider here two situations: the pure neutrino gas (Sec. 3), and a gas containing both neutrinos and antineutrinos (Sec. 4). Throughout the paper, we assume that hard-scattering processes are negligible compared to forward scattering. This is valid provided the energy E of the neutrino satisfies the condition $G_F E^2/(\hbar c)^3 \ll 1$. Forward scattering corresponds to phase-interference effects and hence to neutrino

oscillations. Under these conditions, neutrinos are neither created nor destroyed but are simply transformed from one flavor to another.

For simplicity in what follows, we restrict ourselves to oscillations between electron and muon neutrinos. For vacuum oscillations, relevant parameters are the vacuum mixing angle θ and the mass-squared difference $\Delta = (m_2^2 - m_1^2) c^4$. The effective hamiltonian describing free neutrino propagation is diagonal in the mass-eigenstate basis. However, weak-interaction processes produce and destroy flavor-eigenstate neutrinos. The mass-eigenstate neutrinos ν_1 and ν_2 are related to the left-handed flavor-eigenstate neutrinos ν_{eL} and $\nu_{\mu L}$ by

$$\nu_1 = \nu_{eL} \cos\theta - \nu_{\mu L} \sin\theta \quad , \qquad \nu_2 = \nu_{eL} \sin\theta + \nu_{\mu L} \cos\theta \quad , \tag{1.1}$$

where m_1 and m_2 are the masses of ν_1 and ν_2, respectively.

The vacuum-oscillation period T_Δ is given by

$$T_\Delta = \frac{4\pi E \hbar}{\Delta} \quad . \tag{1.2}$$

For a gas of N_ν neutrinos with a finite energy spread, neutrinos oscillate with various periods. In the limit of negligible interactions, a neutrino (or mixed neutrino-antineutrino) gas with large N_ν exhibits oscillation decoherence, i.e., the net neutrino-flavor content is time independent at late times. This constant asymptotic behavior is independent of initial conditions. For example, suppose one begins with a gas of electron neutrinos. Some will convert into muon neutrinos so that time-varying behavior occurs initially. The flavor content of an individual neutrino at time t is $1 - \sin^2 2\theta \left(1 - \cos\left(2\pi t/T_\Delta\right)\right)$ for ν_e and $\sin^2 2\theta \left(1 - \cos\left(2\pi t/T_\Delta\right)\right)$ for ν_μ. The summation over cosine functions with various periods T_Δ leads asymptotically to a constant function of t if sufficiently many terms are present, i.e., if N_ν is sufficiently large. The ratio of electron neutrinos to muon neutrinos in the limit $t \to \infty$ becomes the constant factor $(1 - \sin^2 2\theta)/\sin^2 2\theta$.

Vacuum behavior dominates provided the neutrino gas is sufficiently dilute, so that interactions due to Z^0 exchange remain unimportant. The dominance is controlled by a dimensionless parameter κ, defined by

$$\kappa = \frac{\Delta}{2\sqrt{2} G_F E n_\nu} \quad ,$$

where $n_\nu = N_\nu/\mathcal{V}$ is the neutrino density. This parameter can be written as the ratio T_ν/T_Δ, where

$$T_\nu = \frac{2\pi\hbar}{\sqrt{2}G_F n_\nu} \tag{1.3}$$

is the time scale associated with neutrino interactions. When the neutrino density is low, κ is large. Neutrino-neutrino forward scattering occurs infrequently compared to a vacuum oscillation period, and the behavior is similar to a non-interacting gas. This region of parameter space is characterized by decoherence.

In contrast, when the neutrino density is large so that $\kappa \ll 1$, neutrino interactions are important. Many neutrino-neutrino interactions occur during a vacuum-oscillation period. Numerical simulations for the pure neutrino gas [8] reveal the existence in this parameter region of a collective mode of the nonlinear dynamics in which the behaviors of individual neutrinos are correlated. Significant numbers of neutrinos oscillate in unison. We refer to this counterintuitive behavior as self-maintained coherence. Self-maintained coherence is also seen in numerical simulations of a gas of neutrinos and antineutrinos [5, 6, 7]. A system consisting initially of electron neutrinos does not decohere. Instead, oscillatory behavior is observed, even at late times. A primary goal of this paper is to obtain an analytical description of self-maintained coherence for $\kappa \ll 1$.

If $m_2 < m_1$ so that $\Delta < 0$ (or, alternatively, if $\Delta > 0$ and $\theta > \pi/2$), then there is a large region in the Δ-θ parameter space for which self-maintained coherence emerges for neutrino oscillations in the early Universe [6]. The behavior begins smoothly, but, due to the expansion of the Universe and the varying electron and positron densities, coherent oscillations emerge at around 100 seconds after the big bang. Self-maintained coherent oscillations may thus have played a role in early-Universe physics.

Throughout the rest of this paper, we work in units with $\hbar = c = 1$.

II. Background Material

A single relativistic neutrino oscillating in vacuum obeys the equation

$$i\frac{d\nu}{dt} = H\nu \quad , \tag{2.1}$$

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where $\nu(t)$ is the two-component flavor wave function

$$\nu = \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} \tag{2.2}$$

with $\nu_e^* \nu_e + \nu_\mu^* \nu_\mu = 1$, and where the effective hamiltonian H is given by

$$H = \frac{m_1^2 + m_2^2}{4E} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\Delta}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \quad . \tag{2.3}$$

The probability for the particle to be an electron neutrino is $\nu_e^* \nu_e$, while that to be a muon neutrino is $\nu_\mu^* \nu_\mu$.

A convenient and standard vector reformulation of the above equations exists [9]. It is useful both for visualization and for numerical simulation of oscillations. Define the vector

$$\vec{v} \equiv \left(\nu_e^* \nu_e - \nu_\mu^* \nu_\mu, 2\mathrm{Re}(\nu_e^* \nu_\mu), 2\mathrm{Im}(\nu_e^* \nu_\mu) \right) \quad . \tag{2.4}$$

Then, the neutrino oscillation equation (2.3) is equivalent to that governing a particle of unit mass and charge moving in a magnetic field $\vec{B}$ given by

$$\vec{B} = \frac{\vec{\Delta}}{2E} \quad , \tag{2.5}$$

where

$$\vec{\Delta} \equiv \Delta \left(\cos 2\theta, -\sin 2\theta, 0 \right) \quad . \tag{2.6}$$

For an antineutrino, Eqs. (2.3)–(2.6) hold if neutrino wave functions are replaced by antineutrino wave functions, i.e., $\nu \to \bar{\nu}$. We denote the corresponding vector for an antineutrino by $\vec{w}$. Throughout this work, we use the expression "magnetic field" to refer to an effective magnetic field, as opposed to a physical one.

For a gas, there is a vector $\vec{v}^j$ for the jth neutrino and a vector $\vec{w}^k$ for the kth antineutrino. The equations governing the self-interacting gas become [5, 10]

$$\frac{d\vec{v}^j}{dt} = \vec{v}^j \times \vec{B}_v^j \tag{2.7}$$

for vectors associated with neutrinos, and

$$\frac{d\vec{w}^k}{dt} = \vec{w}^k \times \vec{B}_w^k \tag{2.8}$$

for antineutrinos. Here, the magnetic fields $\vec{B}_v^j$ and $\vec{B}_w^k$ are given by

$$\vec{B}_v^j = \frac{\vec{\Delta}}{2E^j} - \vec{V}_{\nu\nu} \ , \qquad \vec{B}_w^k = \frac{\vec{\Delta}}{2\overline{E}^k} + \vec{V}_{\nu\nu}^* \quad , \tag{2.9}$$

where the energy of the kth antineutrino is denoted $\overline{E}^k$. An asterisk on a vector indicates a change in sign of the third component. The vacuum contributions to the magnetic fields are the terms in Eq. (2.9) dependent on $\vec{\Delta}$, given in Eq. (2.6). The potential $\vec{V}_{\nu\nu}$ is generated by Z^0 exchange and is given by

$$\vec{V}_{\nu\nu} = \frac{\sqrt{2}G_F}{\mathcal{V}}\left(\langle \vec{v} \rangle - \langle \vec{w}^* \rangle\right) \quad , \tag{2.10}$$

where $G_F \simeq 1.17 \times 10^{-11}\,\mathrm{MeV}^{-2}$ is the Fermi coupling constant, and

$$\langle \vec{v} \rangle = \sum_j \vec{v}^j \ , \qquad \langle \vec{w} \rangle = \sum_k \vec{w}^k \quad . \tag{2.11}$$

In the absence of $\vec{V}_{\nu\nu}$, the first-order differential equations in (2.7) and (2.8) decouple and are linear. The system is then solvable and the solution corresponds to a non-interacting gas in which each neutrino undergoes vacuum oscillatory behavior. When $\vec{V}_{\nu\nu}$ is present, the equations in (2.7) and (2.8) are both coupled and nonlinear. For this reason, we refer to $\vec{V}_{\nu\nu}$ as the nonlinear term. This nonlinearity leads to interesting effects.

Individual neutrinos and antineutrinos are neither created nor destroyed under our assumptions. The equations expressing this,

$$\frac{d\left(\vec{v}^j \cdot \vec{v}^j\right)}{dt} = 0 \ , \qquad \frac{d\left(\vec{w}^k \cdot \vec{w}^k\right)}{dt} = 0 \quad , \tag{2.12}$$

follow from Eqs. (2.7) and (2.8).

Different normalizations of $\vec{v}^j$ are possible. Above, we have chosen $\vec{v}^j \cdot \vec{v}^j = 1$ along with the interpretation that the index j labels individual neutrinos. For this case, the index j ranges from 1 to N_ν, where N_ν is the total number of neutrinos. A second normalization convention follows from noting that neutrinos with the same energy obey the same oscillation equation. One can therefore perform a sum over all vectors of the same energy. With this second normalization, $|\vec{v}^j|$ represents the

number of neutrinos of energy E^j. The index j then ranges over the possible energy values. Similar normalization choices exist for antineutrinos.

For both the above normalization conventions, the total number of neutrinos N_ν and antineutrinos $N_{\bar{\nu}}$ is given by

$$N_\nu = \sum_j |\vec{v}^j| \ , \qquad N_{\bar{\nu}} = \sum_k |\vec{w}^k| \quad . \tag{2.13}$$

The neutrino and antineutrino densities n_ν and $n_{\bar{\nu}}$ can then be obtained by dividing[1] by $\mathcal{V}$, that is, $n_\nu = N_\nu/\mathcal{V}$, $n_{\bar{\nu}} = N_{\bar{\nu}}/\mathcal{V}$.

For numerical purposes the second normalization scheme is more useful. For the mathematical treatment in the current work, we use the first convention with $\vec{v}^j \cdot \vec{v}^j = 1$ and $\vec{w}^k \cdot \vec{w}^k = 1$.

III. The Pure Neutrino Gas

In this section, we analyze self-maintained coherence for a pure neutrino gas. For this system, $N_{\bar{\nu}} = 0$ and the contribution to the neutrino potential becomes

$$\vec{V}_{\nu\nu} = \frac{\sqrt{2} G_F}{\mathcal{V}} \langle \vec{v} \rangle \quad . \tag{3.1}$$

For definiteness, we consider the situation in which N_ν electron neutrinos are placed in the box at time $t = 0$, so that the initial conditions are

$$\vec{v}^j(0) = (1, 0, 0) \quad . \tag{3.2}$$

At $t = 0$, the ratio κ^j of the vacuum term to the neutrino-neutrino term for the jth neutrino is

$$\kappa^j = \frac{\Delta}{2\sqrt{2} G_F E^j n_\nu} \quad . \tag{3.3}$$

When $\kappa^j \ll 1$, the vacuum term is dominated by the neutrino-neutrino interaction term, and self-maintained coherence appears in computer simulations [8]. Neglecting the non-linear term, neutrinos with larger energies oscillate slower and neutrinos with

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[1] Both normalization schemes discussed here can also be modified by multiplying neutrino vectors by $1/\mathcal{V}$. In this situation, neutrino vectors become densities, and $\mathcal{V}$ in the nonlinear term $\vec{V}_{\nu\nu}$ given in Eq. (2.10) must be replaced by 1.

smaller energies oscillate faster. However, a large $\vec{V}_{\nu\nu}$ term boosts slow neutrinos and retards fast neutrinos.

To obtain an analytical solution, we can take advantage of a feature of the motion called alignment [7]: numerical simulation shows that vectors in the nonlinear system point in a common direction when the κ^j are small. Alignment of the jth neutrino implies that the approximation

$$\vec{v}^j(t) \approx \frac{\langle \vec{v} \rangle}{N_\nu} \equiv \vec{r}_v(t) \tag{3.4}$$

is good. Here, $\vec{r}_v(t)$ is the average neutrino vector. This feature suggests we seek an analytical solution for $\vec{r}_v(t)$.

An equation for $\vec{r}_v(t)$ can be obtained by summing over j in Eq. (2.7) and using Eqs. (2.9), (2.11) and (3.1):

$$\frac{d\vec{r}_v}{dt} = \vec{r}_v \times \frac{\vec{\Delta}}{2E_0} \quad , \tag{3.5}$$

where the average inverse energy $1/E_0$ is defined by

$$\frac{1}{E_0} \equiv \frac{1}{N_\nu} \sum_j \frac{1}{E^j} \quad . \tag{3.6}$$

Equation (3.5) shows that the self-maintained coherence in the pure neutrino gas is formally equivalent to the oscillation of a single neutrino in vacuum. Hence, the average neutrino vector undergoes vacuum oscillations with an effective energy E_0.

It is useful for later purposes, when the neutrino-antineutrino gas is considered, to display the solution of Eq. (3.5). The initial conditions for $\vec{r}_v$ are

$$\vec{r}_v(0) = (1, 0, 0) \quad . \tag{3.7}$$

Equation (3.5), when written in components, is

$$\frac{dr_{v1}}{dt} = \left(\frac{\Delta}{2E_0} \sin 2\theta \right) r_{v3} \quad ,$$

$$\frac{dr_{v2}}{dt} = \left(\frac{\Delta}{2E_0} \cos 2\theta \right) r_{v3} \quad ,$$

$$\frac{dr_{v3}}{dt} = -\frac{\Delta}{2E_0} \left(r_{v1} \sin 2\theta + r_{v2} \cos 2\theta \right) \quad . \tag{3.8}$$

These equations simplify in the vacuum-mass-eigenstate basis denoted by $\vec{R}(t)$ and given by

$$R_1 \equiv r_1 \cos 2\theta - r_2 \sin 2\theta \quad , \quad R_2 \equiv r_1 \sin 2\theta + r_2 \cos 2\theta \quad , \quad R_3 \equiv r_3 \quad . \qquad (3.9)$$

In this basis, the equations resemble those in Eq. (3.8) with $\theta = 0$. Thus, $R_1(t)$ is a constant. The equations for R_2 and R_3 combine to give a harmonic-oscillator system. Incorporating the initial conditions (3.7), we find

$$R_1(t) = \cos 2\theta \quad ,$$

$$R_2(t) = \sin 2\theta \cos\left(\frac{\Delta}{2E_0}t\right) \quad ,$$

$$R_3(t) = -\sin 2\theta \sin\left(\frac{\Delta}{2E_0}t\right) \quad . \qquad (3.10)$$

Returning to the flavor basis, we obtain the desired solution:

$$r_1(t) = \cos^2 2\theta + \sin^2 2\theta \cos\left(\frac{\Delta}{2E_0}t\right) \quad ,$$

$$r_2(t) = -\sin 2\theta \cos 2\theta \left(1 - \cos\left(\frac{\Delta}{2E_0}t\right)\right) \quad ,$$

$$r_3(t) = -\sin 2\theta \sin\left(\frac{\Delta}{2E_0}t\right) \quad . \qquad (3.11)$$

Summarizing, the solution in the dense neutrino parameter region is given by Eqs. (3.4) and (3.11). These equations describe self-maintained oscillations. All neutrinos oscillate in unison. Note that perfect alignment is obtained in the limit $\kappa^j \to 0$.

When some κ^j are large, the corresponding neutrinos do not participate in the collective mode. If most neutrinos have small κ^j then self-maintained coherence still occurs but with a smaller amplitude. The criterion for self-maintained coherence for a pure neutrino gas is $\kappa_0 < 1$, where

$$\kappa_0 \equiv \frac{\Delta}{2\sqrt{2}G_F E_0 n_\nu} \quad . \qquad (3.12)$$

We have compared our analytical solution to numerical simulations. Excellent agreement is obtained when all $\kappa^j \ll 1$. Even for the case in Figure 8 of ref. [8], for which 10% of the neutrinos had $\kappa^j > 1$, agreement between the analytical approach and numerical simulations is to about 5% for the oscillation period and the amplitude.

An intuitive understanding of alignment and self-maintained coherence is as follows. Assume that most neutrino vectors are aligned. These vectors point along the average vector $\vec{r}(t)$ and collectively rotate around $\vec{\Delta}$. Consider a particular neutrino with a higher energy than average. Let $\vec{v}$ be its vector. In the absence of the nonlinear term, $\vec{v}$ rotates around $\vec{\Delta}$ at a relatively slow rate. Suppose $\vec{v}$ begins to lag $\vec{r}$. Then, because the nonlinear term is much bigger than the vacuum term, the neutrino experiences a large magnetic field in the direction of $\vec{r}(t)$. Consequently, $\vec{v}$ rotates around $\vec{r}$. After half a period, $\vec{v}$ will have rotated to a position leading the group. Hence, $\vec{v}$ cannot lag behind or otherwise separate from the group.

A similar argument holds for any neutrino with energy lower than average. If $\vec{v}$ begins to lead $\vec{r}$, it experiences a large magnetic field in the direction of $\vec{r}(t)$ and so rotates around $\vec{r}$ rather than $\vec{\Delta}$. It follows that neutrinos with energies different from the average do not rotate around $\vec{\Delta}$ at varying rates but instead stay together in a group. This is alignment. Since the group follows $\vec{r}$, which rotates around $\vec{\Delta}$, oscillatory behavior arises. This is self-maintained coherence.

IV. The Neutrino-Antineutrino System

In this section, we study self-maintained coherence for a dense gas containing both neutrinos and antineutrinos. Numerical simulations reveal that alignment holds separately for neutrinos and antineutrinos [6, 7]. Consequently, the approximations

$$\vec{v}^j(t) \approx \frac{\langle \vec{v}(t) \rangle}{N_\nu} \equiv \vec{r}_v(t) \; , \qquad \vec{w}^k(t) \approx \frac{\langle \vec{w}(t) \rangle}{N_{\bar{\nu}}} \equiv \vec{r}_w(t) \tag{4.1}$$

for the jth neutrino and the kth antineutrino are good. By summing over j and k in Eqs. (2.7) and (2.8), differential equations for $\vec{r}_v$ and $\vec{r}_w$ are obtained:

$$\frac{d\vec{r}_v}{dt} = \vec{r}_v \times \left(\frac{\vec{\Delta}}{2E_0} - \vec{V}_{\nu\nu} \right) \quad , \tag{4.2}$$

$$\frac{d\vec{r}_w}{dt} = \vec{r}_w \times \left(\frac{\vec{\Delta}}{2\overline{E}_0} + \vec{V}_{\nu\nu}^* \right) \quad , \tag{4.3}$$

where

$$\vec{V}_{\nu\nu} = \sqrt{2} G_F \left(n_\nu \vec{r}_v - n_{\bar{\nu}} \vec{r}_w^* \right) \quad . \tag{4.4}$$

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In Eq. (4.2), $1/E_0$ is the average inverse neutrino energy (3.6), and $1/\overline{E}_0$ is the analogous quantity for antineutrinos.

For definiteness, we consider the situation with an equal number of electron neutrinos and antineutrinos placed in the box at time $t = 0$, $n_\nu = n_{\bar{\nu}}$. Then, the initial conditions are

$$\vec{v}^j(0) = (1, 0, 0) \ , \qquad \vec{w}^k(0) = (1, 0, 0) \quad . \tag{4.5}$$

Also, for simplicity we take the antineutrinos to have the same average inverse energy as neutrinos: $\overline{E}_0 = E_0$. This holds, for example, in the more restricted case when the energy distributions of neutrinos and antineutrinos are the same, $\overline{E}^k = E^j$ for all $k = j$.

The symmetry of the initial conditions suggests the ansatz

$$\vec{r}_w(t) = \vec{r}_v(t) \quad . \tag{4.6}$$

It follows that Eqs. (4.2) and (4.3) are equivalent, so it suffices to solve one of the pair to demonstrate consistency of the ansatz. The nonlinear term in Eq. (4.4) only has a third component, $\vec{V}_{\nu\nu} = 2\sqrt{2} n_\nu G_F r_{v3} (0, 0, 1)$. The problem therefore reduces to solving the equations

$$\frac{dr_{v1}}{dt} = \left(\frac{\Delta}{2E_0} \sin 2\theta \right) r_{v3} - 2\sqrt{2} n_\nu G_F r_{v2} r_{v3} \quad ,$$

$$\frac{dr_{v2}}{dt} = \left(\frac{\Delta}{2E_0} \cos 2\theta \right) r_{v3} + 2\sqrt{2} n_\nu G_F r_{v1} r_{v3} \quad ,$$

$$\frac{dr_{v3}}{dt} = -\frac{\Delta}{2E_0} (r_{v1} \sin 2\theta + r_{v2} \cos 2\theta) \tag{4.7}$$

determining the components of average neutrino vector.

These equations again simplify in the vacuum-mass-eigenstate basis $\vec{R}(t)$ given in Eq. (3.9). The vector $\vec{R}$ obeys Eq. (4.7) for $\vec{r}$ with $\theta \to 0$. In what follows, it is convenient to make the further change of variables

$$R_1(t) = y_1(s) \ , \qquad R_2(t) = y_2(s) \ , \qquad R_3(t) = \sqrt{\frac{\kappa_0}{2}} y_3(s) \quad , \tag{4.8}$$

where κ_0 is given in Eq. (3.12) and where

$$s = \mu t \quad , \qquad \mu = \frac{\Delta}{E_0} \sqrt{\frac{1}{2\kappa_0}} \quad . \tag{4.9}$$

The oscillation equations simplify to

$$\frac{dy_1}{ds} = -y_2 y_3 \quad , \tag{4.10}$$

$$\frac{dy_2}{ds} = y_1 y_3 + \frac{\kappa_0}{2} y_3 \quad , \tag{4.11}$$

$$\frac{dy_3}{ds} = -y_2 \quad . \tag{4.12}$$

The initial conditions (4.5) become

$$\vec{R}(0) = \vec{y}(0) = (\cos 2\theta, \sin 2\theta, 0) \quad . \tag{4.13}$$

To proceed, we can take advantage of the conservation of neutrino number, which in the present variables is expressed as

$$y_1^2 + y_2^2 + \frac{\kappa_0}{2} y_3^2 = 1 \quad . \tag{4.14}$$

This equation, which is a consequence of Eqs. (4.10)–(4.13), specifies y_2 in terms of y_1 and y_3. Furthermore, an equation for y_3 in terms of y_1 is obtained by substituting (4.12) into (4.10) and integrating. The above observations determine y_2 and y_3 in terms of y_1 as

$$y_2 = \pm\sqrt{1 - y_1^2 - \kappa_0 \left(y_1 - \cos 2\theta\right)} \quad , \quad y_3 = \pm\sqrt{2\left(y_1 - \cos 2\theta\right)} \quad . \tag{4.15}$$

Specifying the signs corresponds to specifying different stages of the motion, as discussed below (see Eq. (4.23)).

At this point, we need only obtain $y_1(t)$. A differential equation for this variable can be found by differentiating (4.10) with respect to s, using Eqs. (4.11) and (4.12), incorporating Eqs. (4.14) and (4.15), multiplying by dy_1/ds, and integrating once. The integration constant is fixed using Eq. (4.13). This procedure gives

$$\frac{1}{2}\left(\frac{dy_1}{ds}\right)^2 = -V\left(y_1\right) \quad , \tag{4.16}$$

where

$$-V\left(y_1\right) = \left(y_1 - \cos 2\theta\right)\left(1 - y_1^2 - \kappa_0 \left(y_1 - \cos 2\theta\right)\right) \quad . \tag{4.17}$$

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Equation (4.16) is analogous to one describing the motion of a particle of unit mass moving in the potential $V(y_1)$ associated to an anharmonic oscillator with a cubic term.

The motion of y_1 is between $y_1^{\min}$ and $y_1^{\max}$, where

$$y_1^{\min} = \cos 2\theta \ , \qquad y_1^{\max} = x_+ + \cos 2\theta \ . \tag{4.18}$$

The point x_+ determines one of the two zeros of the potential $-V(x + \cos 2\theta) \equiv -x(x - x_+)(x + x_-)$, where

$$x_\pm \equiv \sqrt{1 + \kappa_0 \cos 2\theta + \kappa_0^2/4} \mp \cos 2\theta \mp \kappa_0/2 \ . \tag{4.19}$$

In the region $y_1^{\min} \leq y_1 \leq y_1^{\max}$, it follows that $-V(y_1) \geq 0$, so both sides of Eq. (4.16) are positive.

The solution of Eq. (4.16) is by quadrature in x. Changing the integration variable using $x = x_+ w^2$ gives the implicit solution

$$s = \sqrt{\frac{2}{x_-}} \int_0^{\sqrt{\frac{y_1(s) - \cos 2\theta}{x_+}}} \frac{dw}{\sqrt{(1 - w^2)(1 + q^2 w^2)}} \ , \tag{4.20}$$

where $q^2 = x_+/x_-$. This expression can be inverted for $y_1(s)$ using the sine-amplitude and delta-amplitude Jacobi elliptic functions sn and dn, defined as

$$u = \int_0^{\mathrm{sn}(u,k)} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} \ , \qquad v = \int_0^{\mathrm{dn}(v,k)} \frac{dt}{\sqrt{(1 - t^2)(1 - (1 - k^2)t^2)}} \ . \tag{4.21}$$

We finally obtain

$$y_1(s) = \cos 2\theta + \frac{\sin^2(2\theta)}{x_0} \frac{\mathrm{sn}^2\left(\sqrt{\frac{x_0}{2}}s, \sqrt{\frac{x_+}{x_0}}\right)}{\mathrm{dn}^2\left(\sqrt{\frac{x_0}{2}}s, \sqrt{\frac{x_+}{x_0}}\right)} \ , \tag{4.22}$$

where $x_0 \equiv x_+ + x_- = 2\sqrt{1 + \kappa_0 \cos 2\theta + \kappa_0^2/4}$.

The motion consists of four stages:

$$\begin{array}{llllll}
\text{stage 1:} & y_2 \geq 0 \ , & y_3 \leq 0 \ , & \frac{dy_1}{ds} \geq 0 \ , & \frac{dy_2}{ds} \leq 0 \ , & \frac{dy_3}{ds} \leq 0 \ , \\
\text{stage 2:} & y_2 \leq 0 \ , & y_3 \leq 0 \ , & \frac{dy_1}{ds} \leq 0 \ , & \frac{dy_2}{ds} \leq 0 \ , & \frac{dy_3}{ds} \geq 0 \ , \\
\text{stage 3:} & y_2 \leq 0 \ , & y_3 \geq 0 \ , & \frac{dy_1}{ds} \geq 0 \ , & \frac{dy_2}{ds} \geq 0 \ , & \frac{dy_3}{ds} \geq 0 \ , \\
\text{stage 4:} & y_2 \geq 0 \ , & y_3 \geq 0 \ , & \frac{dy_1}{ds} \leq 0 \ , & \frac{dy_2}{ds} \geq 0 \ , & \frac{dy_3}{ds} \leq 0 \ .
\end{array} \tag{4.23}$$

Since $y_1 \geq \cos 2\theta$, it follows that $y_1 > 0$ for all stages of the motion. During the motion, $y_2^{min} \leq y_2 \leq y_2^{max}$, where $y_2^{max} = \sin 2\theta = -y_2^{min}$.

The motion of $\vec{y}$ is roughly circular about $\vec{\Delta}$. Recall that $\vec{\Delta}$ points along the 1-axis of the vacuum-mass-eigenstate basis. At $t = 0$, $\vec{y} = (\cos 2\theta, \sin 2\theta, 0) = (y_1^{min}, y_2^{max}, 0)$, so $\vec{y}$ points in the direction of the 1-axis of the flavor basis. During stage 1, $\vec{y}$ drops below the 1-2 plane. It then passes to stage 2 when it drops below $\vec{\Delta}$. At the 1-to-2 transition point, $y_1 = y_1^{max}$ and $y_2 = 0$. The vector $\vec{y}$ continues its motion below the 1-2 plane during stage 2, until y_1 and y_2 obtain their minimum values and $y_3 = 0$. In stages 3 and 4, the motion is reversed, except that y_3 is positive so that $\vec{y}$ is above the 1-2 plane. The maximum value of y_1 is again achieved at the 3-to-4 transition, where $\vec{y}$ is above $\vec{\Delta}$ and $y_2 = 0$. The cycle is completed when $\vec{y}$ returns to $\vec{y} = (\cos 2\theta, \sin 2\theta, 0)$. Hence, one entire cycle of motion involves two cycles of y_1. The signs of y_2 and y_3 in Eqs. (4.15) are determined from the second and third columns in Eq. (4.23).

Figures 1a-c display the behavior over two periods of each of the three components of the vector $\vec{r}$ for the case with $\sin^2 2\theta = 0.81$ and $\kappa_0 = 0.1$. The time scale is plotted in terms of the s variable, which is equivalent to the time t measured in units of $1/\mu$. It is convenient to use s so that Δ and E_0 need not be specified (compare with Eq. (4.9)). Extra oscillations appear in the second component, plotted in Fig. 1b. They arise from the projection of the orbit onto the 2 axis. The variables y_1, y_2, y_3 have no such effects. Figure 2 shows the same orbit in a three-dimensional plot.

In general, the half-period of y_1 in s is determined from Eq. (4.20) by setting $y_1 = y_1^{max}$. Since a complete cycle involves two y_1 cycles, one obtains

$$S_{\nu\bar{\nu}} = 4\sqrt{\frac{2}{x_-}} \int_0^1 \frac{dw}{\sqrt{(1 - w^2)(1 + q^2 w^2)}} \tag{4.24}$$

for the period $S_{\nu\bar{\nu}}$ in s. Equation (4.9) then implies that the period $T_{\nu\bar{\nu}}$ in t is

$$T_{\nu\bar{\nu}} = \frac{S_{\nu\bar{\nu}}}{\mu} \quad . \tag{4.25}$$

Hence, the period $T_{\nu\bar{\nu}}$ is of the order of the geometric mean of the time scales associated with the vacuum and nonlinear terms: $T_{\nu\bar{\nu}} \sim \sqrt{T_\Delta T_\nu}$. The motion for the

neutrino-antineutrino gas is thus on the order of $1/\sqrt{\kappa_0}$ times faster than in the pure neutrino case of Sect. 3.

Anther interesting feature of the behavior of the neutrino-antineutrino system is its near planarity, apparent since R_3 is related to y_3 by a factor of $\sqrt{\kappa_0/2}$. The range of R_3 is of order $\sqrt{\kappa_0} \ll 1$. Hence, the bulk of the motion is in the 1-2 plane. Planarity is a feature observed in the numerical simulations of refs.[5, 6, 7]. In the variables $\vec{R}$ or $\vec{r}$, the orbit is similar to a highly eccentric ellipse.

V. Summary

In this paper, we have provided analytical solutions to the nonlinear equations describing the behavior of a gases containing two flavors of neutrinos, both with and without antineutrinos. For a dense pure neutrino gas, the solution is given by Eqs. (3.4) and (3.11), while for a dense neutrino-antineutrino gas the solution is specified by Eqs. (3.9), (4.8), (4.9), (4.15), (4.22) and (4.23).

The behavior of the neutrino-antineutrino gas differs from that of the pure neutrino case. The former is controlled by elliptic functions, whereas the latter is governed by trigonometric functions. Our analytical results agree in detail with prior numerical simulations in the region with $\kappa \ll 1$.

We have demonstrated analytically that self-maintained coherent flavor oscillations occur when the gas density exceeds a critical value, given in terms of the mean neutrino energy and the neutrino masses. Oscillations of this type may have occurred in the early Universe.

Acknowledgements

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FIGURE CAPTIONS

Figure 1. Components of the vector $\vec{r}$ as a function of scaled time, $s = \mu t$, for the case $\sin^2 2\theta = 0.81$ and $\kappa_0 = 0.1$. (a) The component r_1. (b) The component r_2. (c) The component r_3.

Figure 2. The three-dimensional orbit for the case $\sin^2 2\theta = 0.81$ and $\kappa_0 = 0.1$.

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